Modelling Gravitational Events using Semi-analytical Techniques

Denis J Sullivan
(and Paul Chote, Michael Miller)
Victoria University of Wellington, NZ

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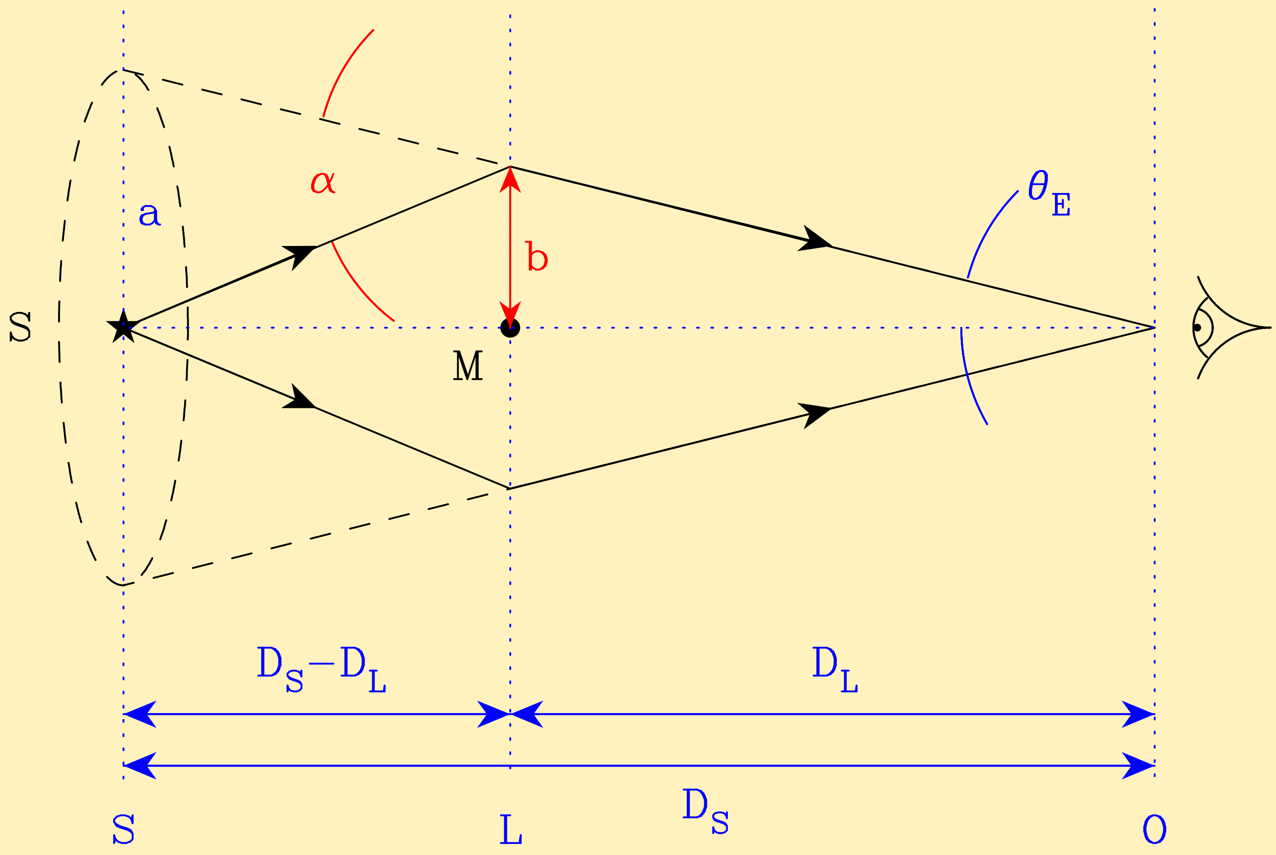
Microlensing basics: the Einstein ring radius

Perfect alignment of a source and lensing mass yields the Einstein ring circular image (which sets the angular scale for microlensing events).

$$\theta_E = \sqrt{\frac{2R_S}{D}} \quad \text{where} \quad R_S = \frac{2GM}{c^2} \quad \text{and} \quad D = \frac{D_L}{D_S - D_L} D_S$$



Microlensing towards the Galactic Bulge

- $R_S \sim 3 \times 10^3$ for $M_L \sim M_\odot$
- $D \sim 8\,\text{kpc} \sim 2.6 \times 10^{20}\,\text{m}$
- Hence: $\theta_E \sim 5 \times 10^{-9}\,\text{rad} \sim 1$ milliarcsec (mas).
- One conclusion: for Galactic events *milliarcseconds* + *not really a lens*, phenomenon should be called *millimiraging*.
- Three useful radii are the Einstein angle projected on to the three relevant planes:

  (a) $r_E = \theta_E D_L$ (lens)

  (b) $\hat{r}_E = \theta_E D_S$ (source)

  (c) $\tilde{r}_E = \theta_E D$ (observer)

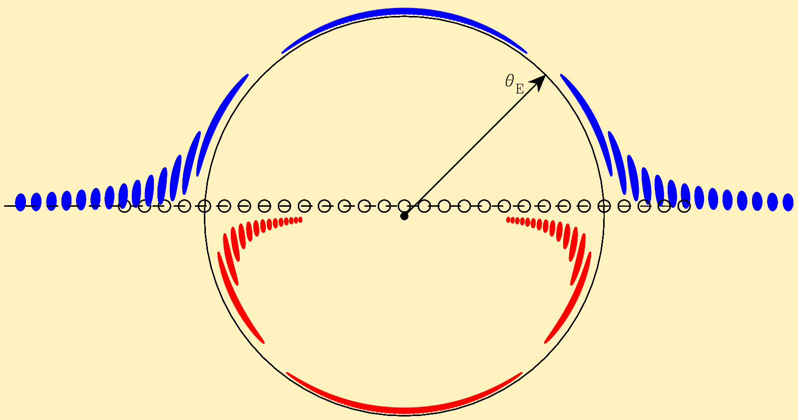
Point source point mass microlensing – single lens

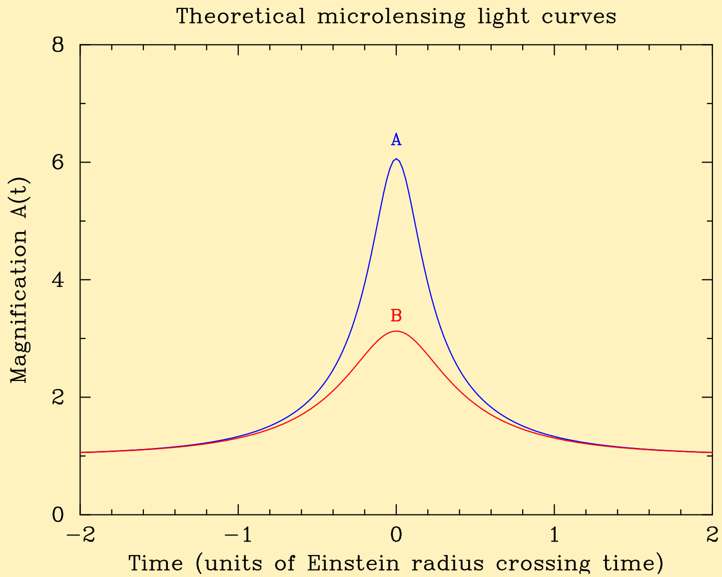
The basic thin lens equation in units of the Einstein ring radius θ_E is:

$u = \theta - \dfrac{1}{\theta}$ where u is source distance; θ is image distance

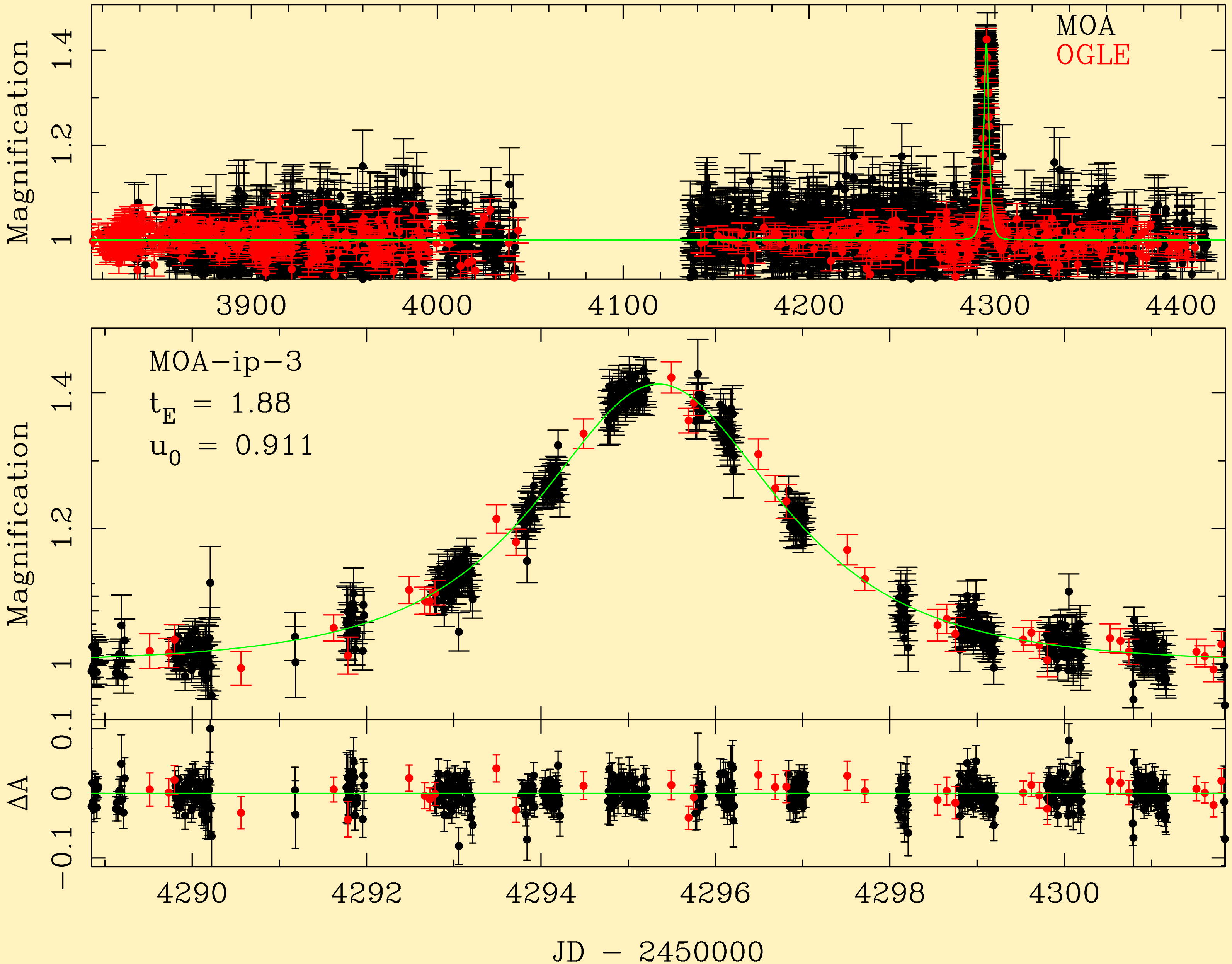
$$A(t) = \sum \left| \frac{\theta d\theta}{u du} \right| = \frac{u^2(t) + 2}{u(t)\sqrt{u^2(t) + 4}}$$ and

$$u^2(t) = u_0^2 + \left(\frac{t - t_0}{t_E} \right)^2 \quad \Longrightarrow$$ the Paczyński light curve

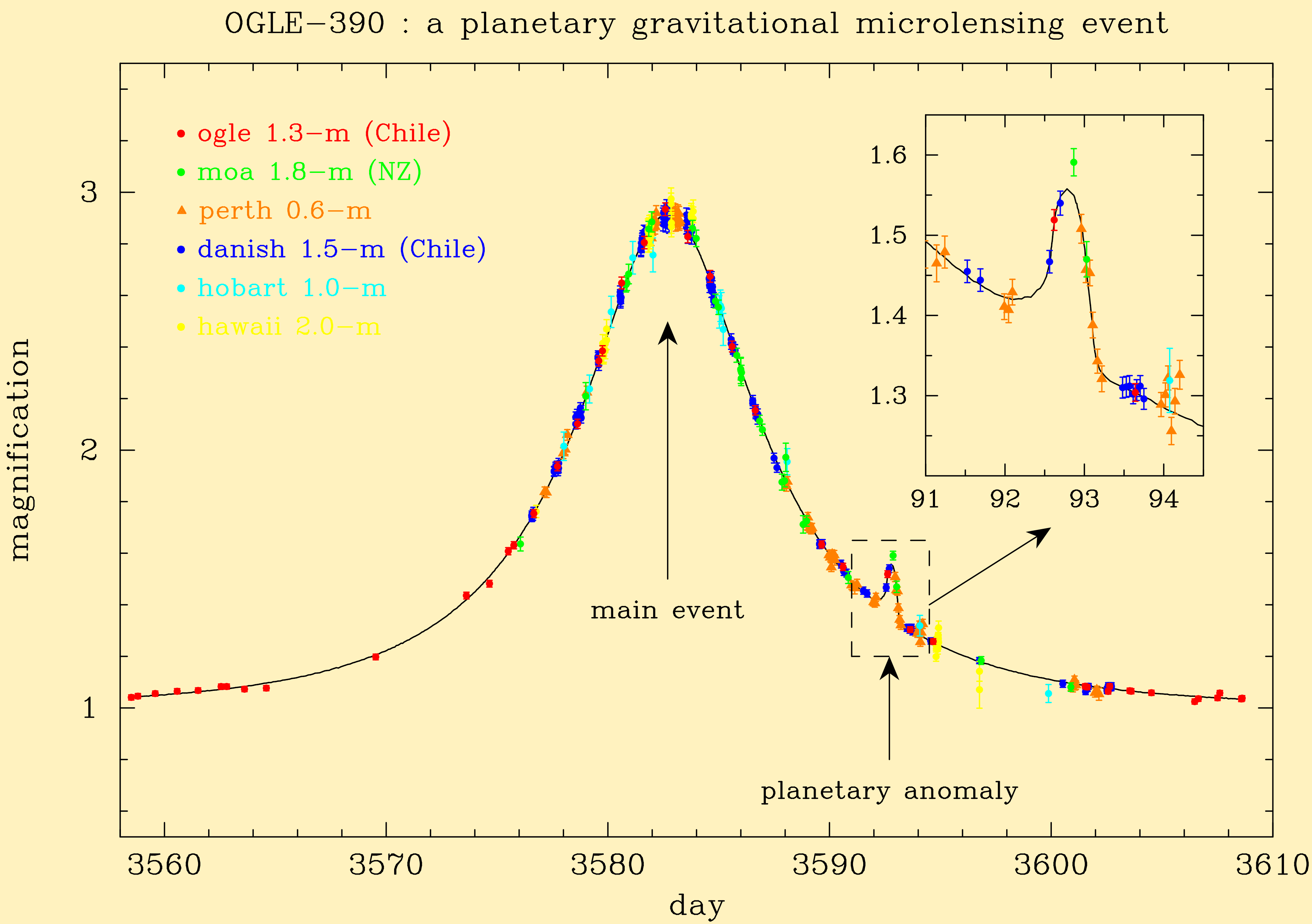




A point lens microlensing event – Sumi et al.



Another one (almost) – Beaulieu et al.



A view of the 1.8 m MOA telescope, Mt John, NZ

Point source point mass lensing – N lenses

- In units of θ_{E} (or r_{E}) for the total lensing mass the basic thin lens equation relating the vector source position $\mathbf{s}$ to the vector *positions* $\mathbf{x}$ of the multiple images is:

$$\mathbf{s} = \mathbf{x} - \sum_{j=1}^{\mathrm{N}} \epsilon_j \frac{\mathbf{x} - \mathbf{r}_j}{|\mathbf{x} - \mathbf{r}_j|^2}$$

where $\mathbf{r}_j$ and ϵ_j are the lens positions and mass fractions respectively.

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- Transforming this 2D equation to the complex plane, $\mathbf{s} \longrightarrow \omega$ and $\mathbf{x} \longrightarrow \mathbf{z}$ we can write this lensing equation in a simpler form

$$\omega = \mathbf{z} - \sum_{j=1}^{N} \epsilon_j \frac{\mathbf{z} - \mathbf{r}_j}{|\mathbf{z} - \mathbf{r}_j|^2} \quad \Longrightarrow \quad \omega = \mathbf{z} - \sum_{j=1}^{N} \frac{\epsilon_j}{\bar{\mathbf{z}} - \bar{\mathbf{r}}_j}$$

and use the complex conjugate version of equation to eliminate $\bar{\mathbf{z}}$.

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- In my view, a similar comment can be made about *the surprising effectiveness of using complex variables to describe microlensing involving more than one lens*

The Lensing Polynomial for N Lenses

- We start with

$$\mathbf{z} - \omega - \sum_{j=1}^{N} \frac{\epsilon_j}{(\bar{\omega} - \bar{\mathbf{r}}_j) + \sum_{k=1}^{N} \epsilon_k / (\mathbf{z} - \mathbf{r}_k)} = 0$$

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- Which can be reorganised as

$$\mathbf{z} - \omega - \sum_{j=1}^{N} \frac{\epsilon_j H_N}{(\bar{\omega} - \bar{\mathbf{r}}_j) H_N + G_N} = 0 \quad \text{where} \quad H_N(z) \sim z^N, G_N(z) \sim z^{N-1}$$

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- Which yields a polynomial in complex $\mathbf{z}$ of order $M = N^2 + 1$

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- This polynomial has M complex roots (not all physical)

Lensing polynomial for 2 lenses

- We start with

$$\mathbf{z}-\omega-\frac{\epsilon_1}{(\bar{\omega}-\bar{\mathbf{r}}_1)+\frac{\epsilon_1}{\mathbf{z}-\mathbf{r}_1}+\frac{\epsilon_2}{\mathbf{z}-\mathbf{r}_2}}-\frac{\epsilon_2}{+(\bar{\omega}-\bar{\mathbf{r}}_2)+\frac{\epsilon_1}{\mathbf{z}-\mathbf{r}_1}+\frac{\epsilon_2}{\mathbf{z}-\mathbf{r}_2}}=0$$

- Which yields a complex polynomial of order 5

$$a_5\mathbf{z}^5+a_4\mathbf{z}^4+a_5\mathbf{z}^3+a_4\mathbf{z}^2+a_1\mathbf{z}+a_0=0$$

- With 5 complex roots (and 3 or 5 physical roots)

Lensing polynomial for 3 lenses

- We start with

$$
\begin{aligned}
\mathbf{z} - \omega \quad & - \quad \frac{\epsilon_1}{(\bar{\varpi} - \bar{\mathbf{r}}_1) + \frac{\epsilon_1}{\mathbf{z} - \mathbf{r}_1} + \frac{\epsilon_2}{\mathbf{z} - \mathbf{r}_2} + \frac{\epsilon_3}{\mathbf{z} - \mathbf{r}_3}} \\
& - \quad \frac{\epsilon_2}{(\bar{\varpi} - \bar{\mathbf{r}}_2) + \frac{\epsilon_1}{\mathbf{z} - \mathbf{r}_1} + \frac{\epsilon_2}{\mathbf{z} - \mathbf{r}_2} + \frac{\epsilon_3}{\mathbf{z} - \mathbf{r}_3}} \\
& - \quad \frac{\epsilon_3}{(\bar{\varpi} - \bar{\mathbf{r}}_3) + \frac{\epsilon_1}{\mathbf{z} - \mathbf{r}_1} + \frac{\epsilon_2}{\mathbf{z} - \mathbf{r}_2} + \frac{\epsilon_3}{\mathbf{z} - \mathbf{r}_3}} \\
& = \quad 0
\end{aligned}
$$

- Which yields a complex polynomial of order 10 with 10 complex roots (and 4, 6, . . . physical roots)

$$
a_{10}\mathbf{z}^{10} + a_9\mathbf{z}^9 + \cdots + a_1\mathbf{z} + a_0 = 0
$$

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- N=3, M=10 : as for N=2, but occasional precision problems (not really fixed by moving to quad precision).
- N=4, M=17 : Definite precision problems and not fixed by moving to quad precision). Solution as follows:

Polynomial root finding procedure

- Set origin of coordinates at the central stellar mass and employ Jenkins-Traub algorithm to find roots for this polynomial.

  (The J-T algorithm determines the roots in increasing order of absolute value (so as to minimise numerical error arising from the polynomial deflation that is used).

- Move origin of coordinates to each of the planetary masses in turn and recalculate polynomial coefficients. Use J-T algorithm to find all the roots for each of these new polynomials.

- Transform all polynomial roots to a common origin system and sort values to obtain increasing absolute values.

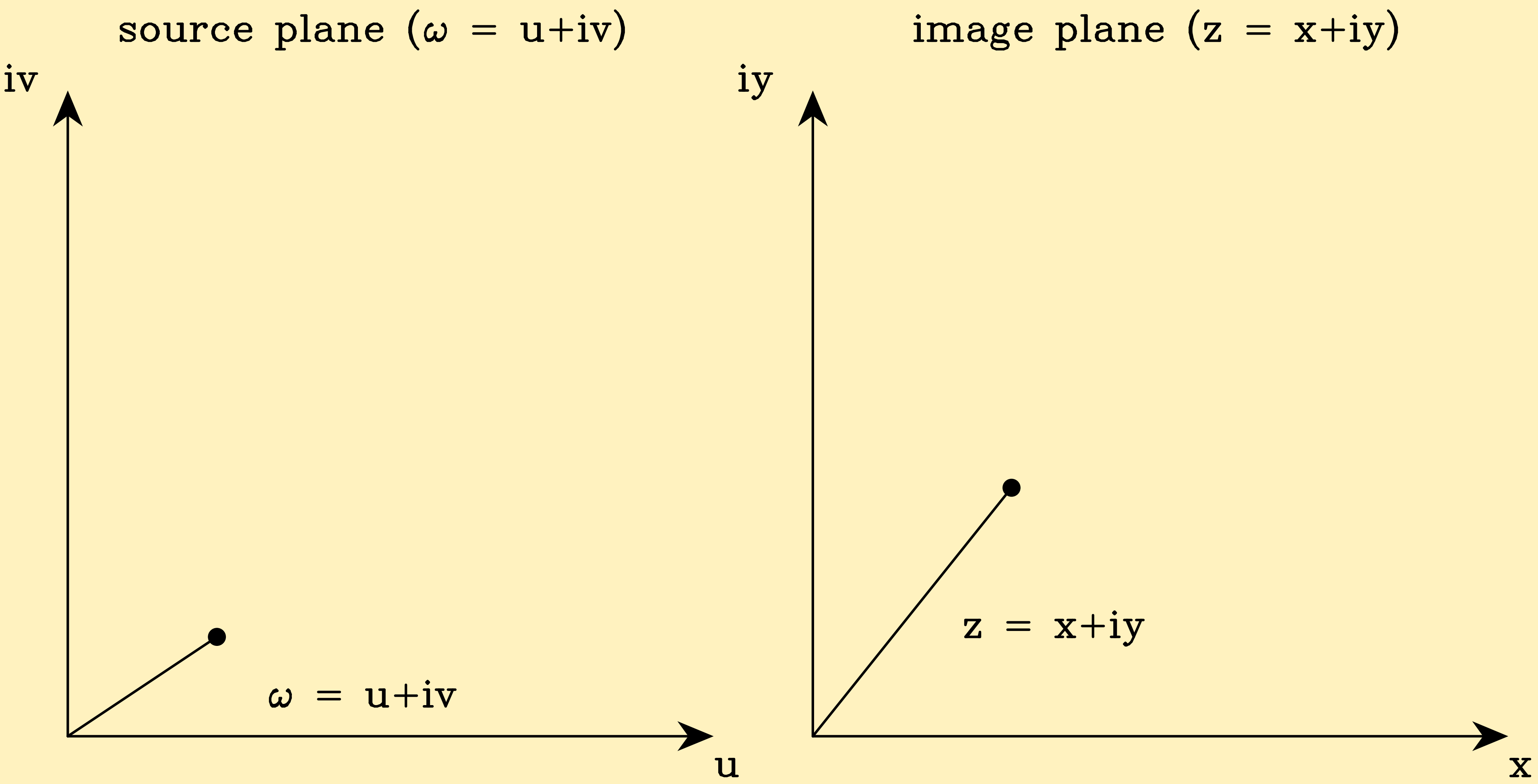
- From grouped "common" value roots, use the lensing equation to determine which roots recover source position and therefore correspond to physical image positions.

Lensing polynomial roots and real images

number of lenses	number of roots	number of images
1	2	2
2	5	3, 5
3	10	4, 6, ...
4	17	5, 7, ...
...	...	...
N	N^2+1	N+1, N+3,...

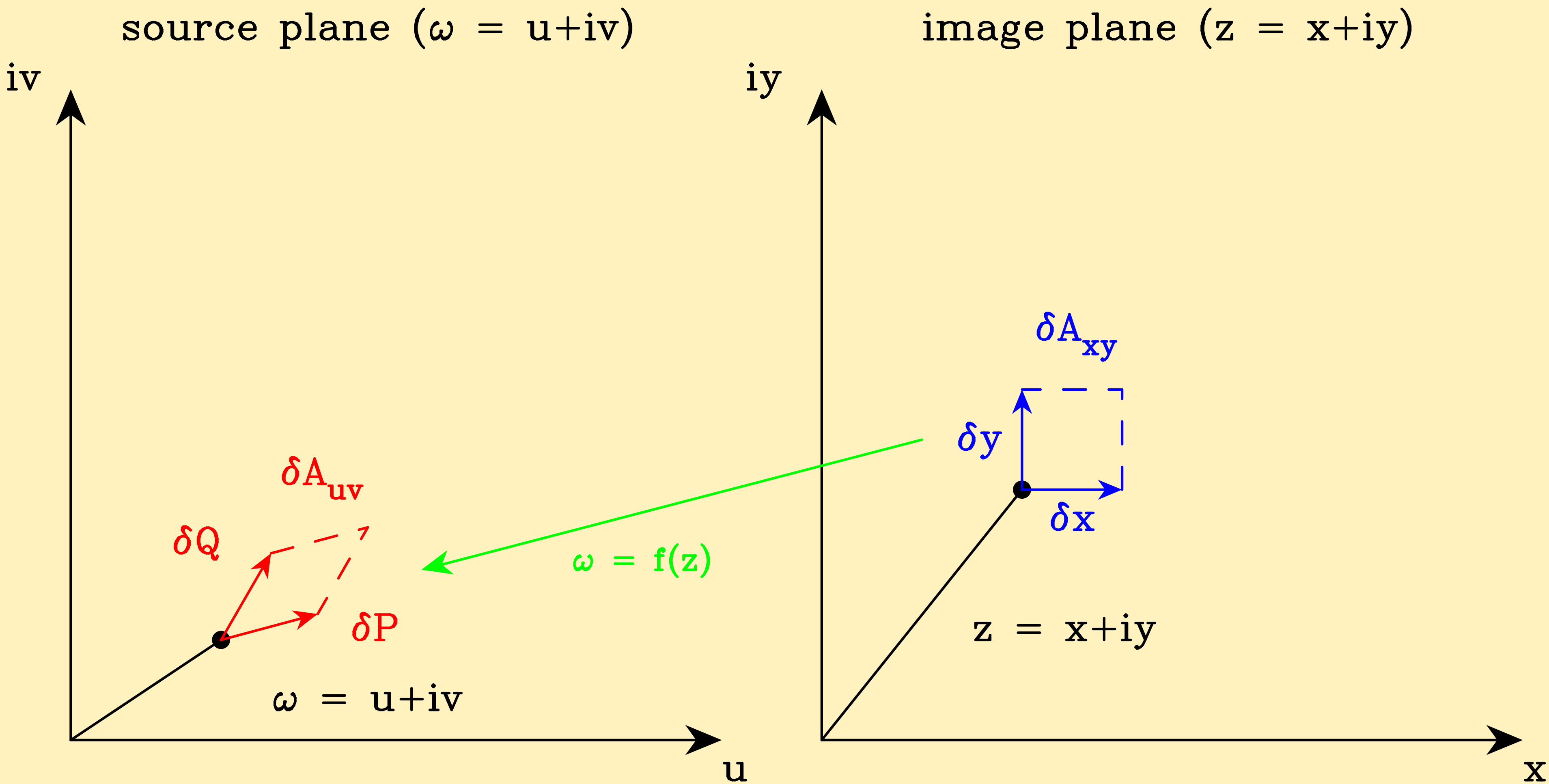
The Source and Image Complex Planes

The algebra is greatly simplified by using complex variables to relate the source position ($\omega = u + iv$) to the image positions ($\mathbf{z} = x + iy$) with transformations $\mathbf{z} = g(\omega)$ and $\omega = f(\mathbf{z})$.



Magnification: Infinitesimal Area Transformation

$$\delta A_{uv} = \delta P_u \delta Q_v - \delta Q_u \delta P_v = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} \right) \delta x \delta y = \frac{\partial (uv)}{\partial (xy)} \delta x \delta y$$



The complex variable Jacobian

- Change variables: $(u, v) \longrightarrow (\omega, \bar{\omega})$ and $(x, y) \longrightarrow (\mathbf{z}, \bar{\mathbf{z}})$
- Jacobian simplifies to

$$J = \frac{\partial(uv)}{\partial(xy)} = \frac{\partial\omega}{\partial\mathbf{z}}\frac{\partial\bar{\omega}}{\partial\bar{\mathbf{z}}} - \frac{\partial\omega}{\partial\bar{\mathbf{z}}}\frac{\partial\bar{\omega}}{\partial\mathbf{z}} = 1 - \left|\frac{\partial\bar{\omega}}{\partial\mathbf{z}}\right|^2 = 1 - \left|\frac{\partial\omega}{\partial\bar{\mathbf{z}}}\right|^2$$

- Hence can evaluate $J(\mathbf{z})$ from expression

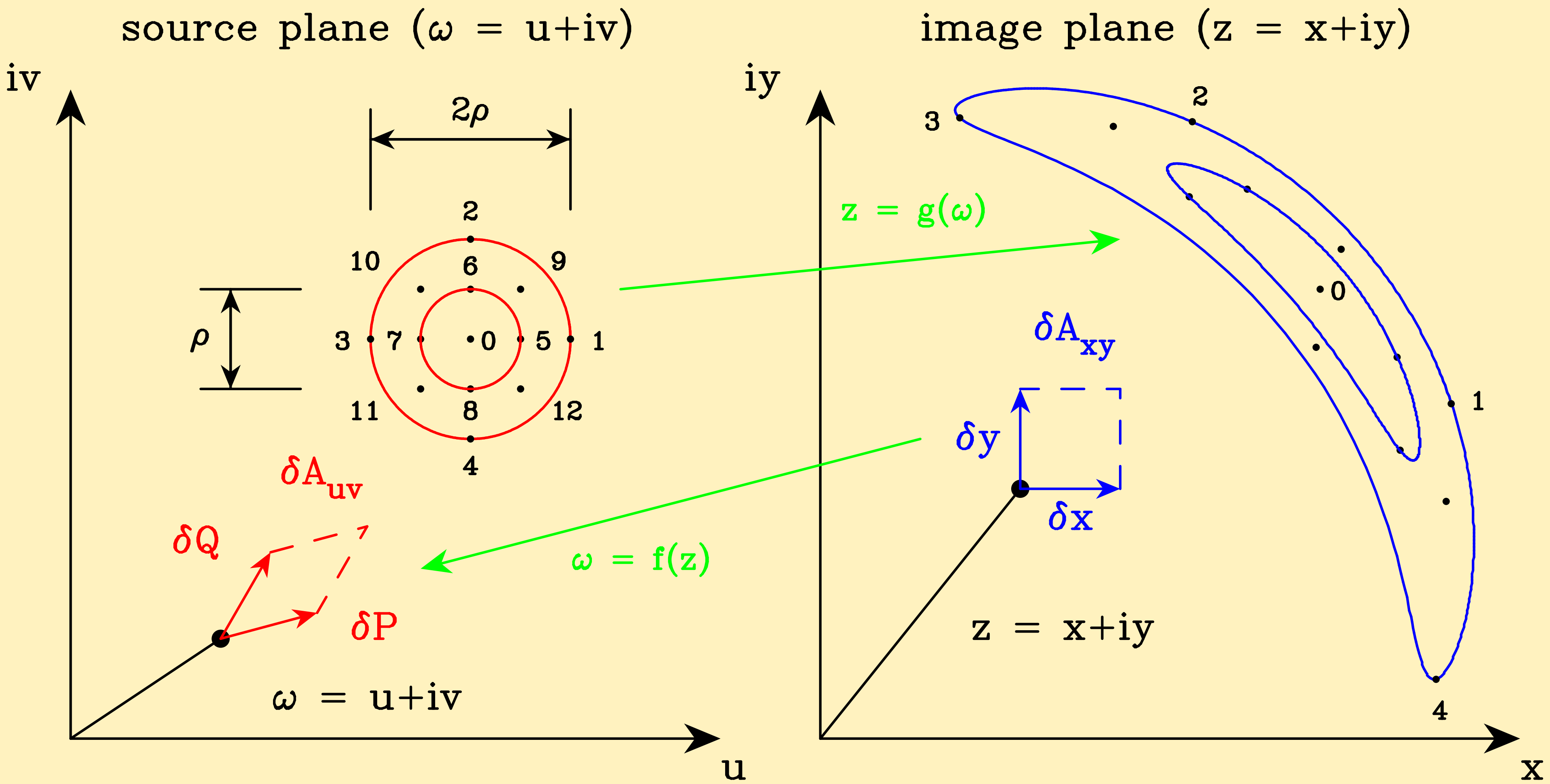
$$J(\mathbf{z}) = 1 - \left|\sum_{j=1}^{N} \frac{\epsilon_j}{(\mathbf{z} - \mathbf{r}_j)^2}\right|^2$$

- Total point source magnification is then sum over N physical images

$$\text{Magn} = \frac{1}{|J(\mathbf{z}_1)|} + \frac{1}{|J(\mathbf{z}_2)|} + \frac{1}{|J(\mathbf{z}_3)|} + \cdots + \frac{1}{|J(\mathbf{z}_\mathrm{N})|}$$

Finite Source Effects

Finite source effects can be included via (a) Quadrapole approximation, (b) Hexadecapole approximation, (c) Discrete equivalent of Green's theorem, or (d) inverse ray tracing in vicinity of image positions.



Determining crtical and caustic curves

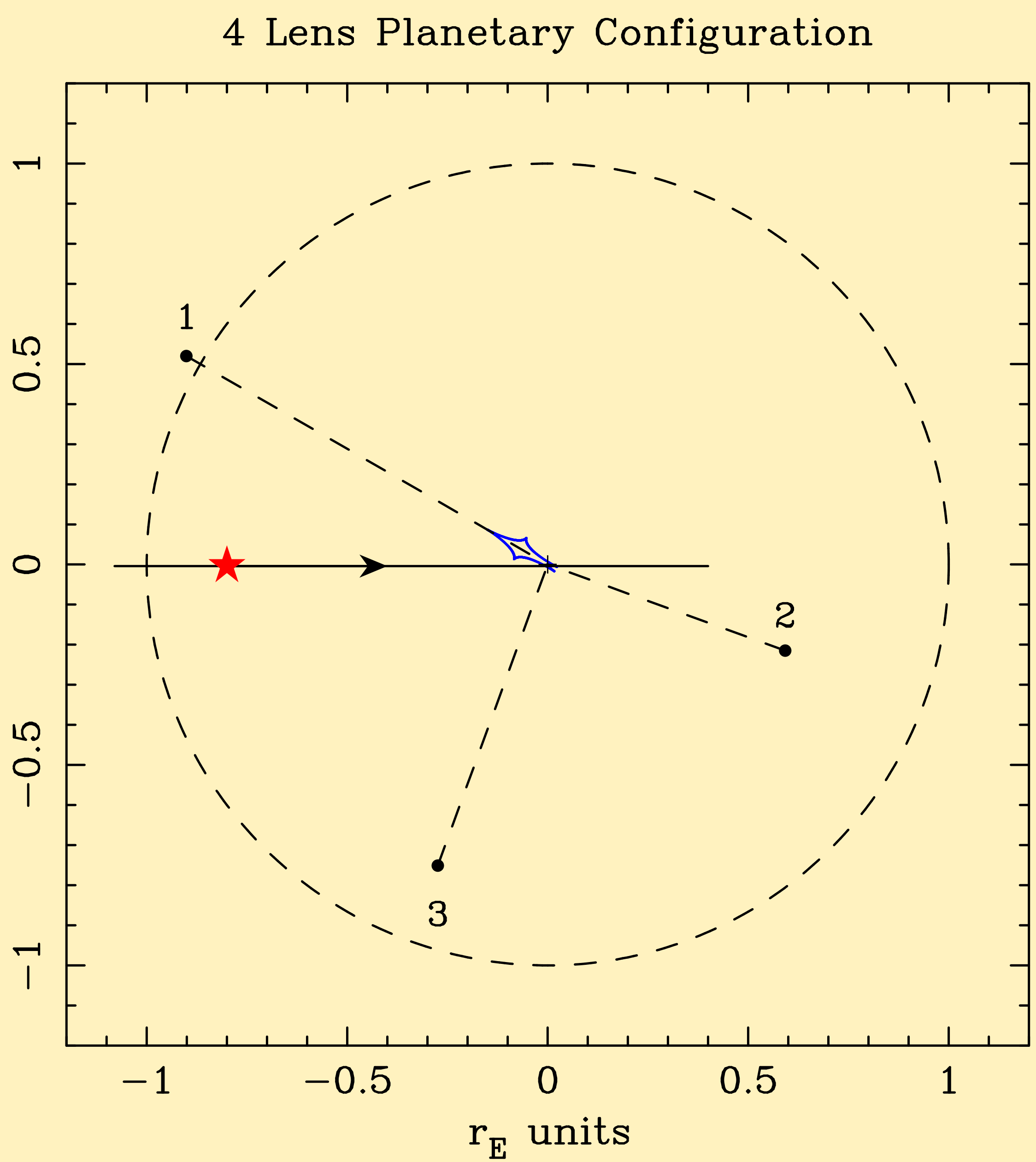
- Require the crtical and caustic curves to characterise the topology of the magnifcation map
- Critical curve positions in image plane identified from $J(\mathbf{z}) = 0$, yielding

$$\sum_{j=1}^{N} \frac{\epsilon_j}{(\mathbf{z} - \mathbf{r}_j)^2} = e^{i\phi} \quad \text{with} \quad 0 \leq \phi \leq 2\pi$$

  This yields a polynomial in $\mathbf{z}$ of order $2N$ with critical curve points determined from polynomial roots. No precision problems as even for $N = 4$ polynomial is *only* of order 8.

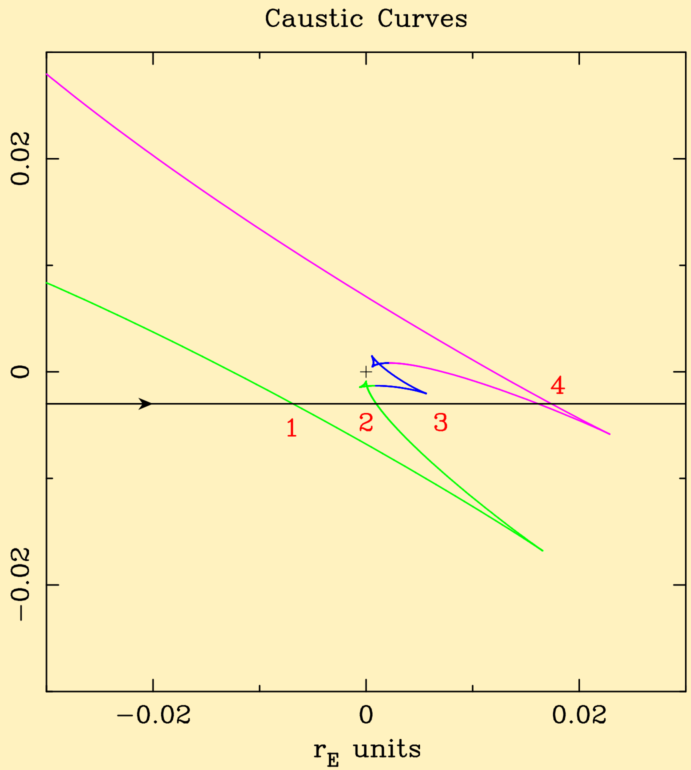
- Use the lens equation to obtain the caustic curves corresponding to positions in the source plane where the amplification is formally infinite.

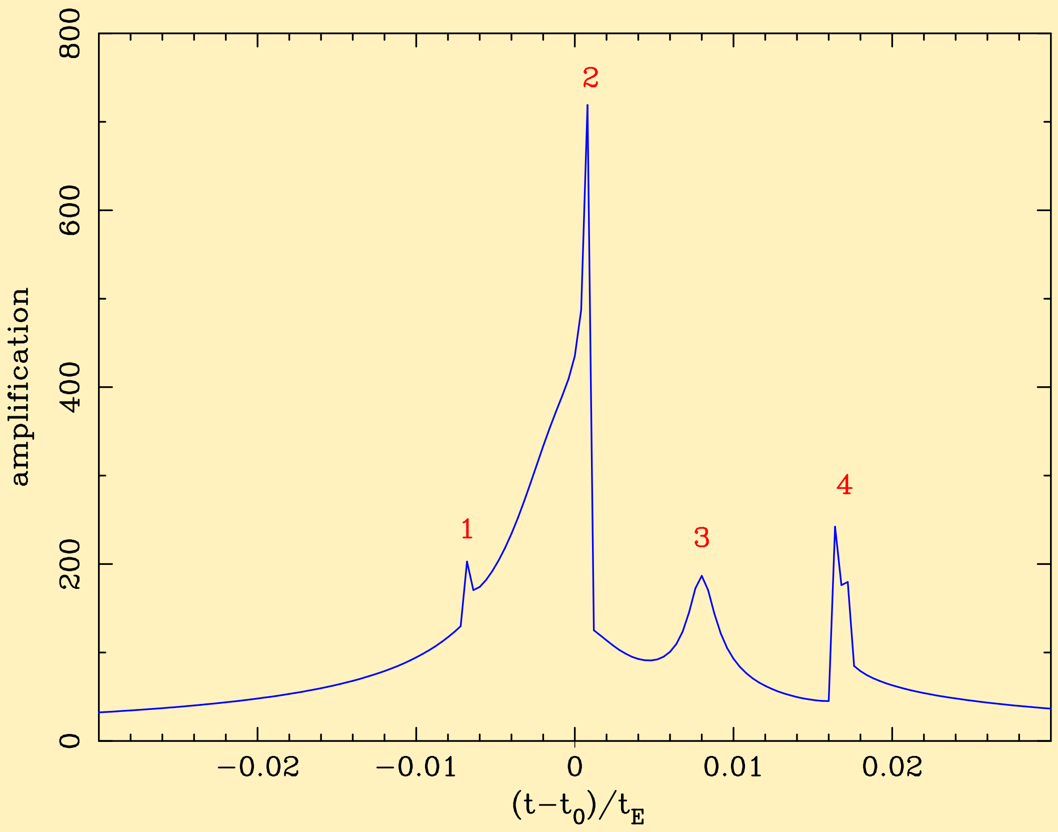
Four lens planetary configuration



A Three lens system ∼ ob06109

$$q_1 = 5.0 \times 10^4 \quad \Big| \quad r_1 = 1.04\, r_E \quad \Big| \quad \theta_1 = 150^\circ$$
$$q_2 = 1.4 \times 10^3 \quad \Big| \quad r_2 = 0.63\, r_E \quad \Big| \quad \theta_2 = -20^\circ$$





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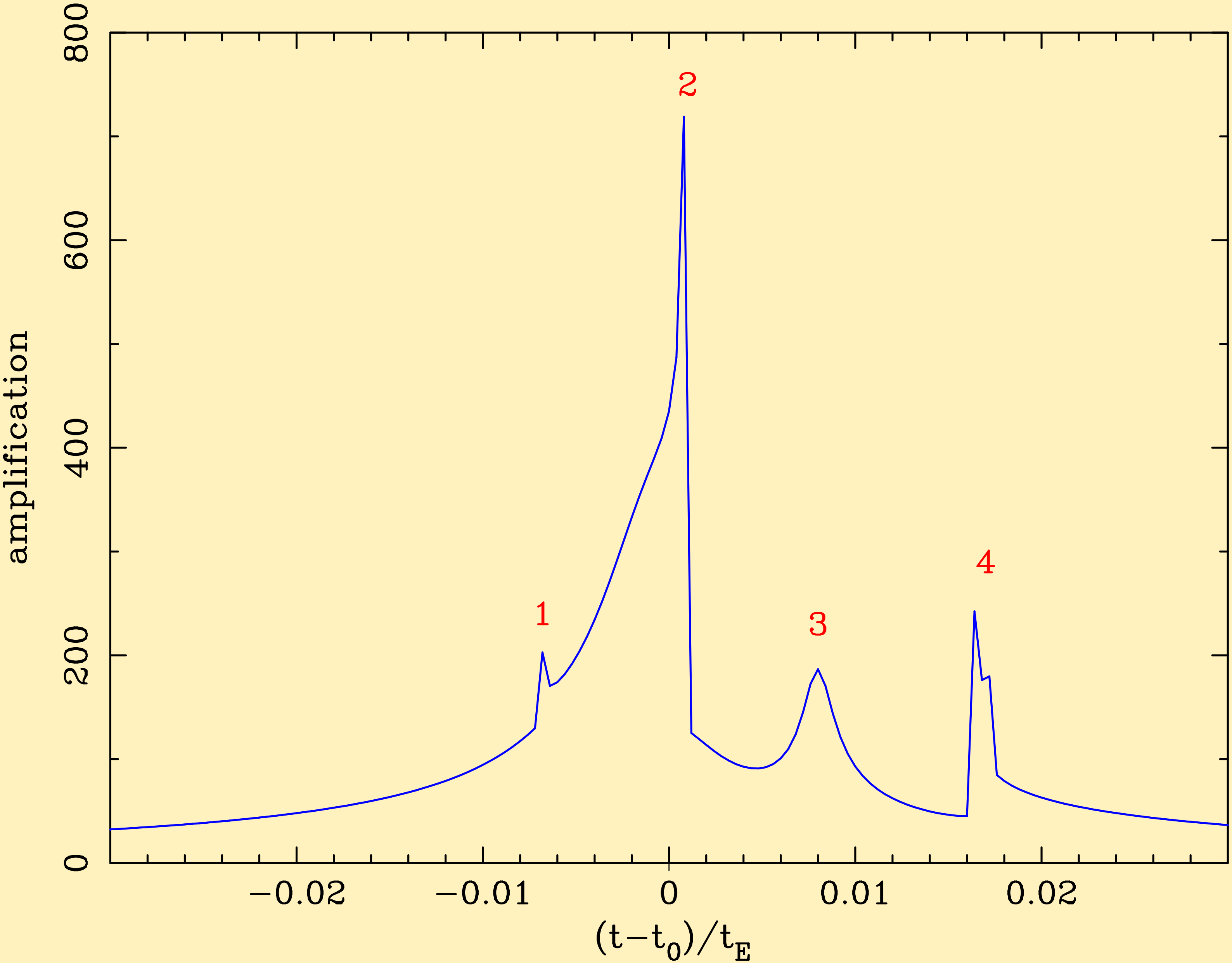
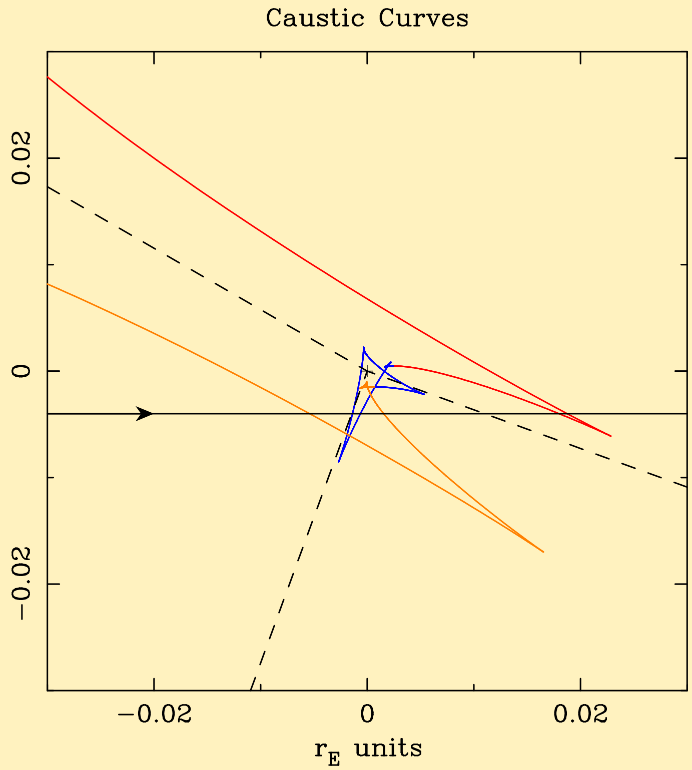
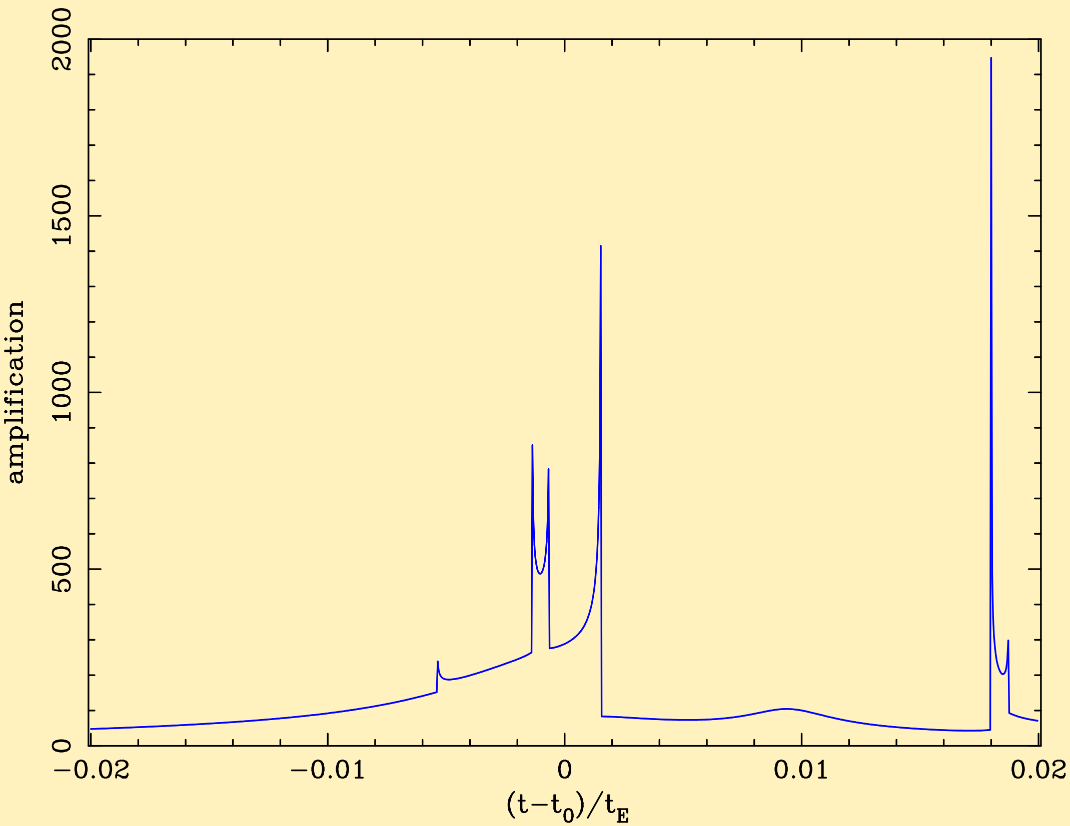


A Four lens system $\sim$ ob06109+

$q_1 = 5.1 \times 10^{-4}$	$r_1 = 1.04\, r_E$	$\theta_1 = 150^\circ$
$q_2 = 1.4 \times 10^{-3}$	$r_2 = 0.63\, r_E$	$\theta_2 = -20^\circ$
$q_3 = 5.1 \times 10^{-4}$	$r_2 = 0.80\, r_E$	$\theta_2 = -110^\circ$





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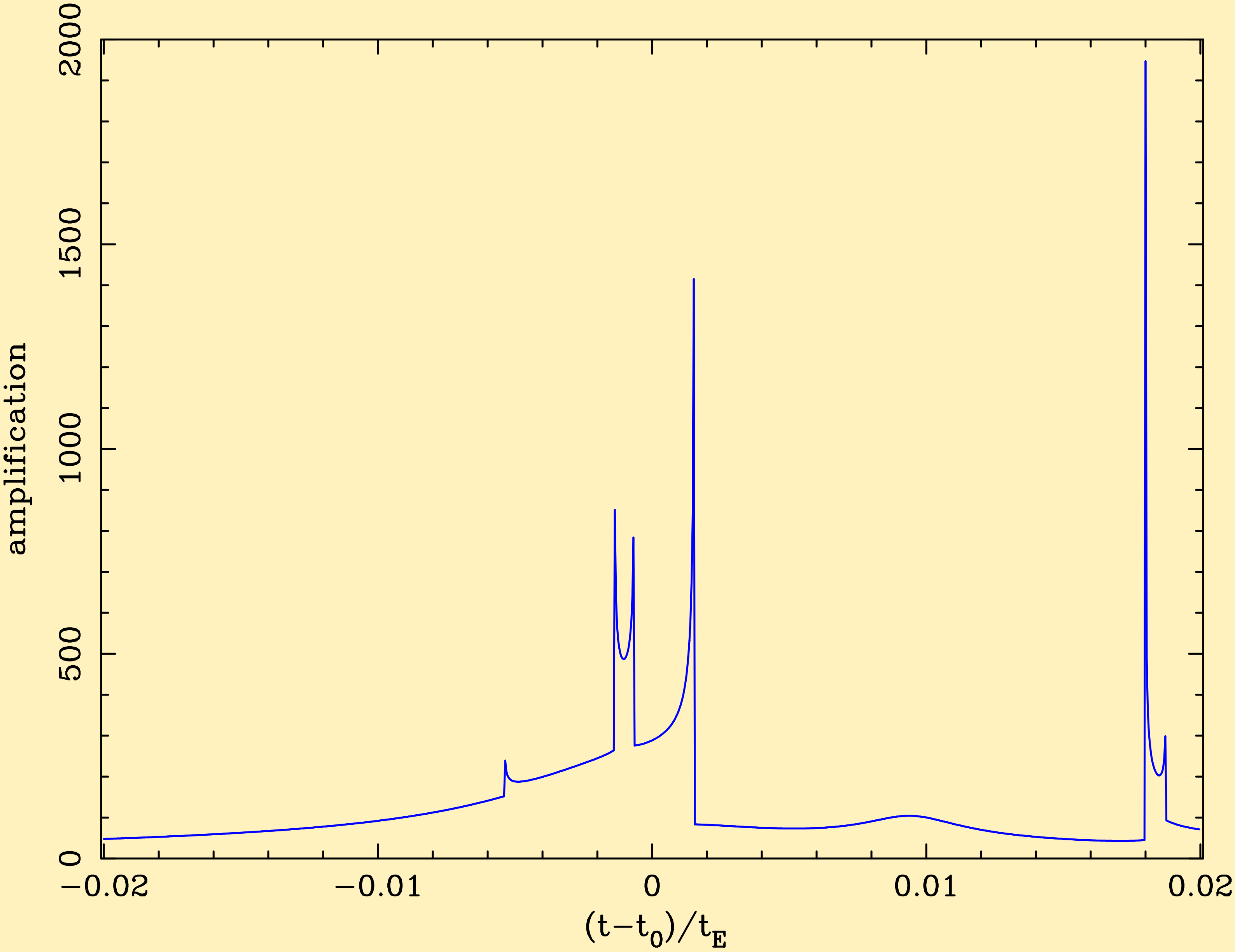


Star-planet systems: caustics $\sim$ add

- For a four lens system the critical curve is determined from

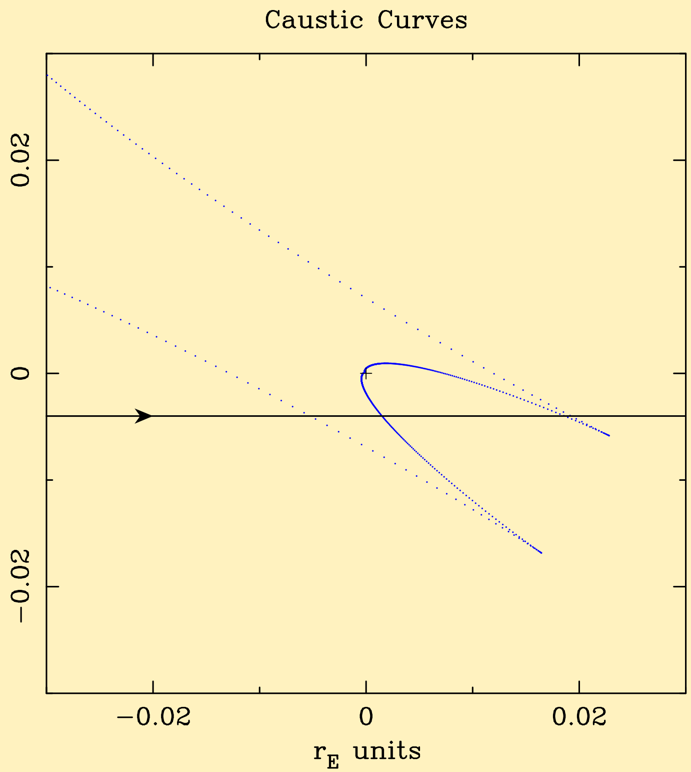
$$\frac{\epsilon_1}{(\mathbf{z}-\mathbf{r}_1)^2}+\frac{\epsilon_2}{(\mathbf{z}-\mathbf{r}_2)^2}+\frac{\epsilon_3}{(\mathbf{z}-\mathbf{r}_3)^2}+\frac{\epsilon_4}{(\mathbf{z}-\mathbf{r}_4)^2}=e^{i\phi}$$

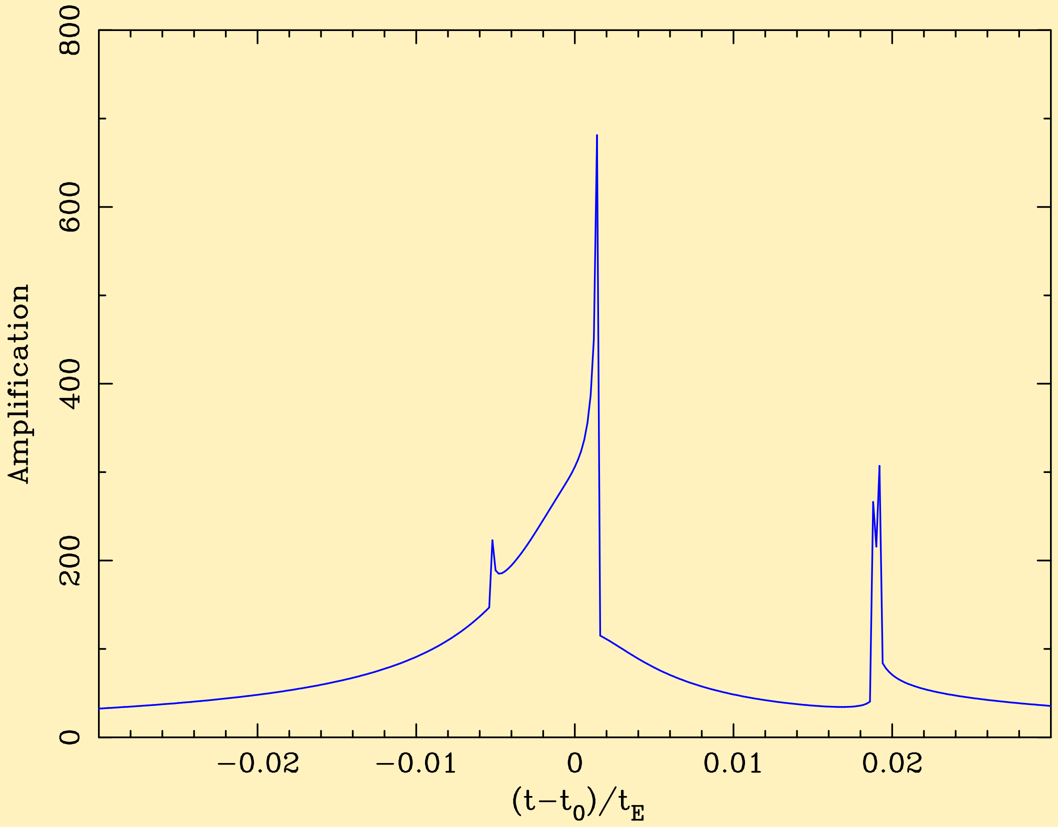
- For a star + three planet system we have $\epsilon_1 \gg \epsilon_2, \epsilon_3, \epsilon_4$, so the interaction between the star and planet i will be dominated by the two-body pair with a critical curve (and caustic) determined from

$$\frac{\epsilon_1}{(\mathbf{z}-\mathbf{r}_1)^2}+\frac{\epsilon_i}{(\mathbf{z}-\mathbf{r}_i)^2}=e^{i\phi}$$

Binary lens components

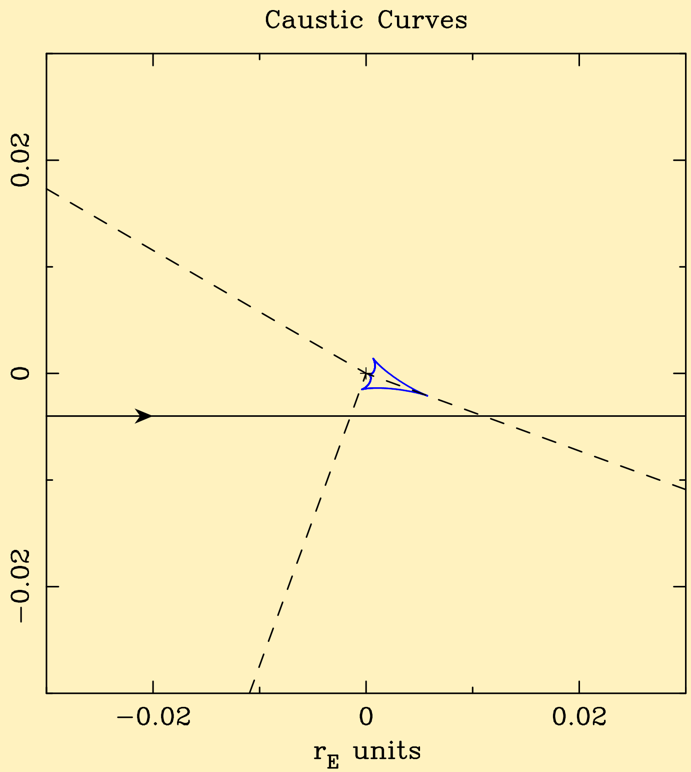
$$q_1 = 5.0 \times 10^{-4} \mid r_1 = 1.04\, r_E \mid \theta_1 = 150^\circ$$

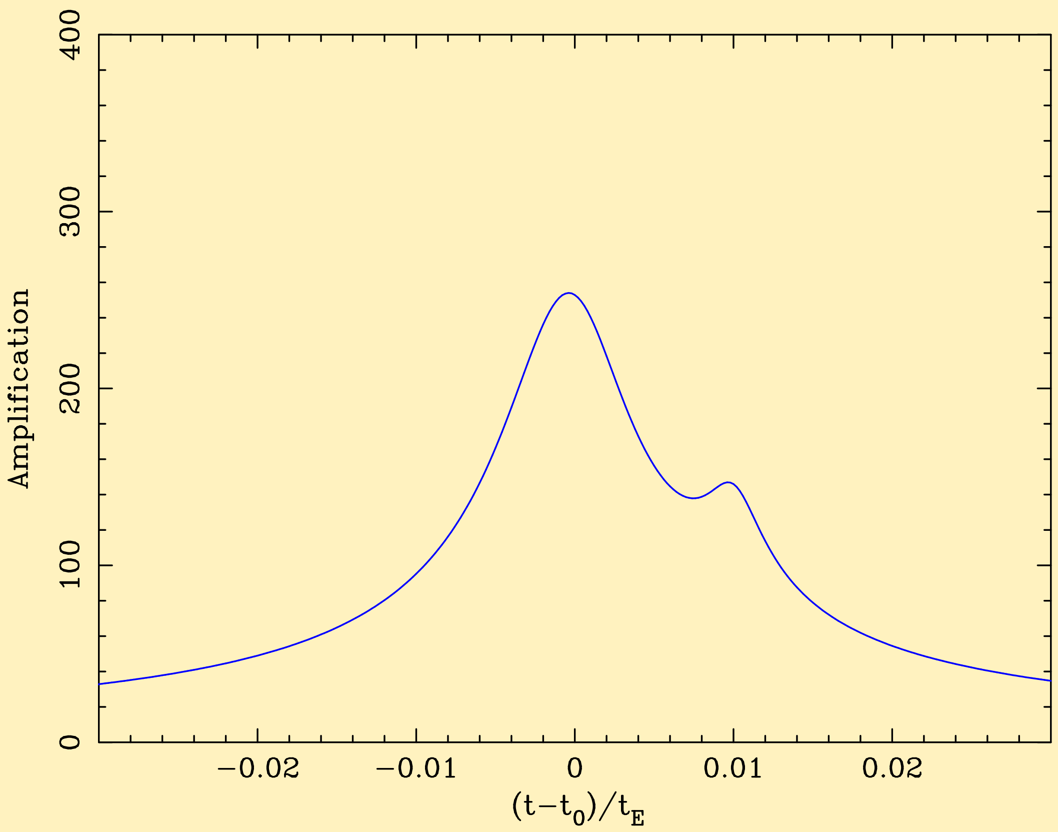




Binary lens components

$$q_2 = 1.4 \times 10^{-3} \;\Big|\; r_2 = 0.63\, r_E \;\Big|\; \theta_2 = -20^\circ$$





Binary lens components

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